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The Geometry of Grand Unification

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Arguments are presented which lead to the conclusion that $SU(3,2)$ is the grand unification gauge group (GUGG). The gauge theory includes all known forces. We incorporated supersymmetry within the framework of the gauge theory and show how the theory may be quantized.

1. THE GRAND UNIFICATION GAUGE GROUP

The geometric formulation (Konopleva and Popov, 1981; Trautman, 1979; Singer, 1980–81; Daniel and Viallet, 1980; Chatelet, 1981; Trautman, 1980; Bleecker, 1981; Herman, 1975, 1977, 1978) of the classical theory of gauge fields requires a connection on a principle fiber bundle, $P(M,G)$. The base space, M, should be space-time. Once a group is specified and the structure constants known, the standard theory provides recipes to follow for construction of the Lagrangian and other physical quantities. The group then (almost) determines the theory:

If $G=U(1)$ we obtain electromagnetic field theory.

If $G=U(1)\times SU(2)$, we obtain the Weinberg–Salam theory of the weak and electromagnetic fields.

If $G=SU(3)$, we obtain quantum chromodynamics: a theory of the strong interactions.

If G is the Lorentz group $SO(3,1)$, we obtain a viable theory of gravitation (Kampfer, 1981; Borchsenius and Mann, 1981).

Since all of the known forces can be obtained by gauging *some* group, is there *one* group which does it all? This is the idea behind grand unification theories (GUTs) (Langlacker, 1981).

The grand unification schemes are intended to unify all the known forces by gauging one group. The first step taken in this direction was by Pati and Salam (1973), who gauged $SU(4)\times SU(4)$. Georgi and Glashow

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(1974) argued that the gauge group should be simple and nominated $SU(5)$ as their candidate. Since then many groups have been suggested as candidates for the grand unification gauge group (Langlacker, 1981). All of these have one shortcoming: they do not yield gravitation. Two things they have in common are having $\mathsf{H} = SU(3)\times SU(2)\times U(1)$ as a subgroup and being compact.

On the other hand many groups have been gauged to obtain something resembling gravitation (Bansobrio, 1980). The one thing all those groups have in common is the subgroup L, the Lorentz group. It is an interesting exercise to compare the list of groups used for unified model building (Slansky, 1981) with the list of those used to obtain gravitation (Bansobrio, 1980). Their intersection is empty. Clearly further analysis is needed. Many particle physicists have expressed the opinion that, in the search for grand unified theories, gravitation can be ignored (to first-order approximation anyway). We do not agree with this point of view. Minkowski showed that Einstein's theory of special relativity implies that space and time must be fused together into one entity: space-time. The four-dimensional space-time is equipped with an indefinite inner product yielding a geometry radically different from Euclid's. The inclusion of gravity in a grand unified theory can be expected to produce changes in the geometry just as dramatic. Indeed, as we will see, this is exactly the case. Let us continue in our search for the grand unification gauge group. If the grand unification concept is valid the required group G will contain H and L. Since it contains L which is noncompact, G will be noncompact. In the symmetry breaking $\mathsf{G} \to \mathsf{H}$ the group H must be the maximal compact subgroup of G in order to avoid having ghosts in the theory (Cremmer and Julia, 1979). Thus we would like to have a real simple Lie Group which contains the Lorentz group and whose maximal compact subgroup is H. Barut and Raczka (1965) have tabulated all the groups containing the Lorentz group, according to the tables in (Gilmore, 1974), there is only one group which satisfies our requirements: $SU(3,2)$, which has 12 noncompact generators and 12 compact generators (Helgason, 1978) (see Table I). The gauge group $SU(5)$ has become the "standard model" since being introduced by Georgi and Glashow (1974). $SU(3,2)$ and $SU(5)$ are related by a "remarkable and important duality between the compact type and the noncompact type" of Lie algebra; see Helgason (1978, p. 235). This duality is known to physicists as the Weyl unitary trick (Gilmore, 1974). The Lie algebra $SU(3,2)$ consists of matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ {}^t\bar{Z}_2 & Z_3 \end{pmatrix}$$

with Z_1, Z_3 skew Hermitean of order 3 and 2, respectively; $\operatorname{Tr} Z_1 + \operatorname{Tr} Z_3 = 0$. To get $SU(5)$ we replace ${}^t\bar{Z}_2$ by $-{}^t\bar{Z}_2$.

TABLE I. Simple Groups Containing the Lorentz Group (Barut and Raczka, 1965) and Their Maximal Compact Subgroups (Gilmore, 1974)

$SO(p,q)$ $p \geqslant 3,\ p \geqslant q \geqslant 1$	$SO(p)\times SO(q)$
$SU(p,q)$ $p \geqslant 3,\ p \geqslant q \geqslant 1$	$SU(p)\times SU(q)\times U(1)$
$U(p,q)$ $p \geqslant 3,\ p \geqslant 3,\ p \geqslant q \geqslant 1$	$U(p)\times U(q)$
$USp(2p,2q)$ $p \geqslant 3,\ p \geqslant q \geqslant 1$	$USp(2p)\times USq(2q)$
$SL(p,c)\ p \geqslant 2$	$SU(p)$
$SL(p,R)\ p \geqslant 4$	$SO(p)$

Since $SU(3,2)$ preserves the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

and $SU(5)$ preserves the form with all positive 1's, $SU(5)$ is called the compact analog of $SU(3,2)$.

All of Georgi and Glashow's arguments which lead "inescapably to the conclusion that $SU(5)$ is the gauge group of the world" are equally as valid for $SU(3,2)$, *except* their requirement that the group be compact. For example, $SU(3,2)$ (1) is of rank 4, (2) allows complex representations, (3) contains H, and (4) predicts a value of $\sin^2\theta_w$ which is in agreement with expected values.

If, indeed, "we cannot unify weak and electromagnetic interactions independently of strong interactions," why should one expect to be able to exclude gravitation from a truly unified theory? Including gravitation requires including the Lorentz group which forces G to be noncompact. Surely this requirement overshadows Georgi and Glashow's reasons for looking at only compact Lie groups. Gauging $SU(3,2)$ will yield all the known forces.

Following Helgason, one possible parameterization of $su(3,2)$ is

$$\begin{matrix} iy_{11} & x_{12}+iy_{12} & x_{13}+iy_{13} & x_{14}+iy_{14} & x_{15}+iy_{15} \\ -x_{12}+iy_{12} & iy_{22} & x_{23}+iy_{23} & x_{24}+iy_{24} & x_{25}+iy_{25} \\ -x_{13}+iy_{13} & -x_{23}+iy_{23} & iy_{33} & x_{34}+iy_{34} & x_{35}+iy_{35} \\ x_{14}-iy_{14} & x_{24}-iy_{24} & x_{34}-iy_{34} & iy_{44} & x_{45}+iy_{45} \\ x_{15}-iy_{15} & x_{25}-iy_{25} & x_{35}-iy_{35} & -x_{45}+iy_{45} & -i(y_{11}+y_{22} \\ & & & & +y_{33}+y_{44}) \end{matrix}$$

The corresponding infinitesimal generators are

$$Y_{ij} = \frac{\partial}{\partial y_{ij}} \text{ (above matrix)}|_{y_{ij}} = 0$$

$$X_{ij} = \frac{\partial}{\partial x_{ij}} \text{ (above matrix)}|_{x_{ij}} = 0$$

The generators of $SU(3)$ are the X_{ij}, Y_{ij}, $i, j \leqslant 3$, $i \neq j$, and $Y_{11} - Y_{33}$, $Y_{22} - Y_{33}$. The generators of $SU(2)$ are the X_{ij}, Y_{ij} $(i, j \geqslant 4)$. The X_{ij} alone generates $SO(3,2)$ the de Sitter group. The X_{ij} $(i, j \leqslant 4)$ generate $SO(3,1)$, the Lorentz group. Thus the X_{ij} $(i, j \leqslant 3)$ are simultaneously generators of the $SU(3)$ and $SO(3,1)$. This mixing is essential since the group $SU(3,2)$ is simple. The diagonalized matrices are Y_{11}, Y_{22}, Y_{33}, and Y_{44}. To obtain the standard diagonalizations, we need to change the diagonal elements of the basis to

$$\lambda_3 = \frac{1}{2}[Y_{11} - Y_{22}] = \begin{pmatrix} \frac{i}{2} & & & & \\ & -\frac{i}{2} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{3}(Y_{11} + Y_{22} - 2Y_{33}) = \begin{pmatrix} \frac{i}{3} & & & & \\ & \frac{i}{3} & & & \\ & & -\frac{2i}{3} & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

$$t_3 = \frac{1}{2} Y_{44} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \frac{i}{2} & \\ & & & & -\frac{i}{2} \end{pmatrix}$$

Requiring that the fourth diagonal operator be orthogonal to all the above via the Killing form, we obtain

$$y=\begin{pmatrix} -\frac{i}{3} & & & & \\ & -\frac{i}{3} & & & \\ & & -\frac{i}{3} & & \\ & & & \frac{i}{2} & \\ & & & & \frac{i}{2} \end{pmatrix}$$

According to Weinberg (1979), the coupling constants for the strong, weak and electromagnetic forces are related by

$$g_s^2 \operatorname{Tr} \lambda_3^2 = g^2 \operatorname{Tr} t_3^2 = g'^2 \operatorname{Tr} y^2$$

For the $SU(3,2)$ model, we compute

$$g_s^2(-\tfrac{1}{2}) = g^2(-\tfrac{1}{2}) = g'^2(-\tfrac{5}{6})$$

$$g_s^2 = g^2 = \tfrac{5}{3}g'^2$$

hence agreeing with Weinberg's heuristic derivations.

The $SU(3)\times SU(2)\times U(1)$ content of $SU(3,2)$ is exactly the same as $SU(5)$; hence in the $SU(3,2)$ theory we have

$$\sin^2\theta_w = \tfrac{3}{8}$$

at the grand unification energy. Since renormalization is done *after* the breaking to H, the coupling constants at ordinary energies are obtained in the standard way exactly as in the $SU(5)$ theory. Since the $SU(5)$ theory agrees with experiment, so does the $SU(3,2)$.

We have imbedded the Lorentz algebra L into $SU(3,2)$. The classic paper about such imbeddings is O'Raifeartaigh (1965). So naturally the question arises: What does O'Raifeartaigh say about this embedding?

The Lie algebra $su(3,2)$ is semisimple, in fact, simple, hence the radical S of $su(3,2)$ reduces to the zero element. Since $S=0$, $P\cap S=0$. This is O'Raifeartaigh's case (iv). He shows that L must be imbedded in a simple Lie algebra which we already have since $su(3,2)$ is simple.

He goes on to say: "No algebra belonging to case (iv) has been proposed in the literature cited ... case (iv) deserves some further investigation," which is exactly what we are doing.

2. THE GEOMETRY AND SYMMETRY BREAKING

The geometrical setting of a principle fiber bundle will be denoted as $P \underset{\pi}{\overset{G}{\to}} M \cong P/G$, where P is the total space M is the base space, π is the projection, and the fibers are all diffeomorphic to G. The breaking of the symmetry from G to H changes the picture to

$$P \underset{\pi_2}{\overset{H}{\to}} P/H \underset{\pi_1}{\overset{G/H}{\to}} M$$

Thus the breaking induces a new fiber bundle $P/H(G/H, M)$ with base M and fiber the homogeneous space G/H.

This is basically, the Higgs mechanism (Chaohao, 1981). The Higgs field is a Yang–Mills field with gauge group H on the principle fiber bundle $P \underset{\pi_2}{\overset{H}{\to}} P/H$. The bundle $P/H \underset{\pi_2}{\overset{G/H}{\to}} M$ then describes the Higgs particles. If we go to tumbling gauge theories and have a further breaking to a subgroup $H_i < H$ we have the symmetry-breaking diagram

$$P \underset{\pi_i}{\overset{H_i}{\to}} P/H_i \underset{\pi_{2,i}}{\overset{H/H_i}{\to}} P/H \underset{\pi_1}{\overset{G/H}{\to}} M$$

The picture drawn so far has a serious defect: The base space M is put in by hand. This goes counter to our entire program of naturality. If we cannot remedy the situation, this entire line of thinking should be scrapped. After making two observations we will be able to proceed.

Observation I. From the mathematical viewpoint, perhaps the most important lesson learned from quantum mechanics is the fundamental role that complex numbers play in our description of nature. Geometrically this leads to consideration of the complexification of the tangent space of a real manifold or all the way to complex manifolds (Wells, 1979).

Observation II. In the geometric setting of soldered gauge theories (Giachetti, et al., 1982) one looks for a subgroup $K < G = SU(3,2)$ such that G/K is locally isomorphic to the base space M. Thus we need a subalgebra $K' < G'$ such that G'/K' is isomorphic to $T_x M$. Thus K' must be 20 dimensional for M to be four dimensional. Furthermore, for the dynamics of relativity we need that the Lorentz group L be contained in K'. There is no subalgebra of G' satisfying these requirements.

We are now in a quandry. Gravitation is a soldered gauge theory (Trautman, 1982) so soldering *must* work *if* we have the correct group. $SU(3,2)$ has too much going for it to be so easily abandoned.

Here observation I comes into play. Suppose we allow the base space to be a complex space time can we then find a subalgebra $K' < G'$ satisfying our requirements? Now, miraculously, the answer is yes. What we want left is the last column of the representation in Section 1; i.e., we want $Q = G'/K'$ to be generated by $X_{15}, Y_{15}, X_{25}, Y_{25}, X_{35}, Y_{35}, X_{45}, Y_{45}$ so we take K' to be the subalgebra generated by everything else. K then turns out to be $SU(3,1)\times U(1)$ and another symmetric space enters the picture: $M = SU(3,2)/SU(3,1)\times U(1)$. Returning to the analogy of $SU(3,2)$ with $SU(5)$ we Euclideanize everything obtaining $SU(5)/SU(4)\times U(1) = \mathbb{P}_4(\mathbb{C})$ which shows why projective spaces have provided good models for field theory.

The Killing form on Q has the correct signature to be a space-time since X_{45} and Y_{45} are generators of $SU(2)$, hence compact, their scalar product is positive definite while the other generators are in the noncompact sector hence negative definite.

Only one more thing to check we have the decomposition

$$G' = K' \oplus Q$$

In order to utilize the results of soldering (Giachetti et al., 1982) we must have that this decomposition is reductive, i.e., that

$$[K', Q] \subset Q$$

An easy calculation shows this is indeed the case.

Now the idea of soldering is to identify M with a section of an associated fiber bundle and use the connection on the PFB to determine the geometry of the section hence the geometry of M. The details of this program will be carried out in a future publication.

We return to the principle fiber bundle $P(M, G)$. Given a representation ρ: $G \to GL(V)$ with V a vector space, we can define an action of G on $P \times V$ by

$$(p, f)\cdot g = (p\cdot g, \rho(g^{-1})f)$$

If

$$k\colon P\times V \to E = (P\times V)/G = P\times_G V$$

is the natural projection onto the quotient of the action of G, the set E has a natural vector bundle structure with typical fiber V.

According to Trautman (1979), a particle of type ρ, interacting with the gauge field is described by a section of the bundle associated to P by ρ. The standard Higgs field corresponds to $V=G'$ (the Lie algebra of G) and $\rho = ad$.

The standard wisdom (Konopleva and Popov, 1981; Trautman, 1979; Singer, 1980–81; Daniel and Viallet, 1980; Chatelet, 1981; Trautman, 1980; Bleecker, 1981; Hermann, 1977, 1975, 1978) would lead us to write the Lagrangian of a particle of type ρ with a field as

$$\mathsf{L}_1(\omega,\phi)=\int_M \|\Omega\|^2+\|D_\omega\phi\|^2 \tag{1}$$

where Ω is the curvature of the connection ω and ϕ is a section of the bundle E as above.

We will later see reasons for believing that the standard wisdom is wrong. The correction necessary will be shown to be very minor—just a reinterpretation of the symbol ϕ in (1).

Also, some theorists would include another term $-m^2\|\phi\|^2$ in the Lagrangian. When we get to quantization, we will require that $\|\phi\|=1$ so we omit this term now (it is unnecessary).

In order to write down the Lagrangian (1), we require a representation of $G=SU(3,2)$. In this formalism, *any* vector space and *any* representation is allowable. Since each representation leads to a different Lagrangian and presumably describes a different particle, we have an infinite number of particles describable by this theory. This freedom to choose arbitrary representations seems too lax and very unnatural.

As Trautman (1982) points out, this freedom of choice in representation spaces is one of the biggest differences between the standard (generalized) Higgs–Yang–Mills gauge theories and gauge theories of gravitation and is one of the biggest stumbling blocks in the (gravitation included) grand unification schemes. Which representations are admissible? The number of particles and their properties should be dictated by the theory not introduced by whim. Thus somehow the number of admissible representation spaces should be limited naturally. The original Higgs description uses the adjoint representation of G on its Lie algebra G$'$. This is a natural representation. We extend this requirement of naturality: the representation spaces must occur naturally in the theory.

One last requirement will fix the vector spaces: these spaces must naturally admit some interpretation of the symmetries and antisymmetries known to occur for wave functions. That is, V must be a subspace of a graded Lie algebra.

There is only one vector space which meets these requirements: $\Lambda(M, \mathrm{G}')$, the space of G' valued forms on M:

$$\Lambda(M,\mathrm{G}') = \bigoplus_{p=0}^{n} \Lambda^p(M,\mathrm{G}') \tag{2}$$

where n = dimension of M.

$\Lambda(M,\mathrm{G}')$ is a graded Lie algebra. The sections ϕ in (1) are cross sections of $\Lambda(M,\mathrm{G}')$, i.e., Lie algebra valued forms. See, e.g., Bleecker (1981) for details. The Lie Algebra G' has a natural inner product which we can extend to $\Lambda(M,\mathrm{G}')$.

This requirement that the sections ϕ be in $\Lambda(M,\mathrm{G}')$ satisfies Trautman's objections and makes the theory compatible with his standard treatment of general relativity as a gauge theory since we already have a soldered gauge theory.

The gauge group G acts on P and this action induces an action (a representation) of G on $\Lambda^p(M,\mathrm{G}')$. Thus there is no need to separately introduce a representation ρ as in the "standard wisdom." The required representations come out of the geometry. They need not be put in by hand.

Definition (Trautman, 1979). The connection form ω and the $\Lambda(M,\mathrm{G}')$ valued form ϕ are said to be *compatible* iff

$$D_\omega \phi = 0 \tag{3}$$

Theorem. If ω and ϕ are compatible, and if

$$D_\omega {*\Omega} = 0 \tag{4}$$

then (ω, ϕ) is a critical point of the Lagrangian (1).

Proof. Write

$$L = L_1 + L_2$$

$$L_1 = \int_M \|\Omega\|^2$$

$$L_2 = \int_M D_\omega\phi \Lambda {*D_\omega\phi}$$

Then (4) implies that the variation of L_1 with respect to ω is zero:

$$\delta_\omega L_1 = 0$$

see Atiyah (1979) while $\delta_\omega \cdot L_2 \cdot \gamma = \int_M D_\nu\phi\Lambda^* D_\omega\phi + D_\omega\phi\Lambda^* D_\nu\phi$ so (3) implies $\delta_\omega L_2 = 0$; hence (3) and (4) together imply

$$\delta_\omega L = 0$$

The variation with respect to ϕ is

$$\int_\phi L \cdot \psi = \int D_\omega\psi\Lambda^* D_\omega\phi + \int D_\omega\phi\Lambda^* D_\omega\psi$$

so (3) implies that

$$\delta_\phi L = 0 \qquad \blacksquare$$

3. QUANTIZATION

Arnold (1967) was the first to note that cohomology conditions were important in quantization. Other workers have introduced other cohomologies in the setting of geometric quantization and field theory (Rawnsley, 1979; Lichnerowicz, 1980). This is the motivation for what follows.

Associated with the graded Lie algebra $\Lambda(M, \mathrm{G}')$, we have the diagram

$$\Lambda^0 \xrightarrow{\delta} \Lambda^1 \xrightarrow{\delta} \Lambda^2 \xrightarrow{\delta} \Lambda \xrightarrow{\delta} \Lambda^4 \tag{5}$$

where δ is some differential operator, yet to be determined. The diagram (5) is a complex iff $\delta \circ \delta = 0$. If $\delta = d$, we get the standard de Rham complex. If $\delta = D_\omega$, the diagram (5) is not a complex since $D_\omega \circ D_\omega \neq 0$ in general.

If however we consider the operator $\delta\theta = -[\omega, \theta]$ we can make (5) into a complex (Ruchti, 1975). To make (5) a complex, we require

$$[\omega, [\omega, \theta]] = 0 \tag{6}$$

So for which ω does (6) hold? We use the graded Jacobi identity (Bleecker, 1981, p. 36).

For $\phi \in \Lambda^i(M, \mathrm{G})$, $\psi \in \Lambda^j(M, \mathrm{G})$, $\rho \in \Lambda^k(M, \mathrm{G})$ we have

$$[\psi, \phi] = -(-1)^{ij}[\phi, \psi] \tag{7}$$

$$(-1)^{ik}[[\phi, \psi], \rho] + (-1)^{kj}[[\rho, \phi]\psi] + (-1)^{ji}[[\psi, \rho], \phi] = 0 \tag{8}$$

In (8) let $\phi = \rho = \omega$ since $\omega \in \Lambda'(M, \mathrm{G}')$ we obtain

$$(-1)[[\omega, \psi], \omega] + (-1)^j[[\omega, \omega], \psi] + (-1)^j[[\psi, \omega], \omega] = 0 \tag{9}$$

But by (6), the first and third terms of (9) vanish so (9) reduces to

$$[[\omega,\omega],\psi]=0 \tag{10}$$

Since this is to hold for all ψ, we must have

$$[\omega,\omega]=0 \tag{11}$$

To satisfy (11), ω must be a multiple of one generator of $SU(3,2)$, e.g., $\omega=\omega_0 e_i$, where e_i is one of the generators of $SU(3,2)$ and ω_0 is a 1-form.

Clearly, further conditions must be placed on the ω—perhaps we should demand that they be connection 1-forms on an appropriate bundle.

The sections of $\Lambda(M,\mathrm{G}')$ can be completed to form a Hilbert space (with indefinite metric).

In this Hilbert space we have a duality mapping (*) which is the extension to Lie algebra valued forms of the standard Hodge star operator (Bleecker, 1981).

If we view quantization as the requirement that only the eigenstates of a set of operators corresponding to the observables can appear, then we must require that the sections be eigenforms of all the operators which appear naturally. Thus we are regarding the quantum numbers as the spectrum of an appropriate complex.

We already have the Hodge * operator which has eigenvalues $+1$ on even forms and -1 on odd forms. Thus we must have that ϕ is odd (fermion) or even (boson).

Other operators appear from the complex ($\Lambda^r=\Lambda^r(M,\mathrm{G}')$):

$$\begin{array}{ccc}
\Lambda^0 & \xrightarrow{*_0} & \Lambda^4 \\
\delta_\omega\downarrow & & \uparrow\delta_\omega \\
\Lambda^1 & \xrightarrow{*_1} & \Lambda^3 \\
\delta_\omega\downarrow & & \uparrow\delta_\omega \\
\Lambda^2 & \xrightarrow{*_2} & \Lambda^2 \\
\delta_\omega\downarrow & & \uparrow\delta_\omega \\
\Lambda^3 & \xrightarrow{*_3} & \Lambda^1 \\
\delta_\omega\downarrow & & \uparrow\delta_\omega \\
\Lambda^4 & \xrightarrow{*_4} & \Lambda^0
\end{array} \tag{12}$$

where $\delta_\omega\theta = -[\omega, \theta]$, and ω is required to satisfy (11). If $\theta \in \Lambda^r$, we do a diagram chase around the two closed loops and sum:

$$\Delta_\omega\theta = \delta_\omega{}^*\delta_\omega{}^*\theta + {}^*\delta_\omega{}^*\delta_\omega\theta$$

For the complex (12) this is the analog of the Laplace Beltrami operator.

So we see that the observables of the particle ϕ will be the eigenvalues of the operators Δ_{ω_i}

$$\Delta_{\omega_i}\phi = \lambda_i\phi \tag{13}$$

for ω_i satisfying (11).

If, in (5) we take $\delta = d$ the quantization condition (13) becomes

$$\Delta\phi = \lambda\phi$$

Taking $\lambda = -m^2$ this reads

$$\Delta\phi = -m^2\phi \tag{14}$$

Because of the indefinite metric on the base space (14) is the Klein–Gordon equation. Thus mass is a quantum number of an interaction which involves gravitation which is where mass should come from. Clearly the spectrum of (14) is continuous as the mass spectrum should be.

The spectrum of (13) is basically the spectrum of the matrix representing e_i and so will be discrete. Noether's theorem tells us to that each generator of $SU(3,2)$, there is a conserved quantity, (13) tells us how to quantize that quantity. The eigenvalues in (13) are the squares of the eigenvalues of these matrices. Again they are negative definite

$$\Delta_{\omega_i}\phi = -\gamma_i^2\phi$$

and the γ_i are the quantum numbers of the particles.

The possible supersymmetry operators (Witten, 1982) in this setting are

$$Q_0 = d + {}^*d^*$$

$$Q_\omega = -[\omega, -{}^*[\omega, {}^*$$

and

$$Q_0 - Q_\omega = D_\omega + {}^*D_\omega{}^*$$

where ω satisfies the condition (11).

Other connections with previous work should be noted:

(1) The real part of the Lie algebra $su(3,2)$ is the de Sitter algebra $so(3,2)$.

(2) Penrose's theory of twistors[1] is based on the Lie group $SU(2,2)$: we are increasing the number of space dimension by 1.

(3) The chronometric theory of I. E. Segal (1976) is based on analysis of $O(3,2)$ and $SU(2,2)$. Evidently the $SU(3,2)$ theory will require an extension of Segal's work.

(4) F. Gürsey (1981) has investigated Hermitian symmetric spaces in other models.

(5) J. W. Moffat (1981) has proposed a theory of gravitation using the groups $U(i, j)$ but he does not specify what i and j should be.

4. DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

$SU(3,2)$ has many features which indicate that it is the proper group to gauge for grand unification. First, it is the *unique* simple group containing the Lorentz group whose maximal compact subgroup is $SU(3)\times SU(2)\times U(1)$. Second, $SU(3,2)$ can be soldered to a complex space-time. Soldering is the most natural, if not the only, way in which the geometry on the bundle can influence the geometry on the base.

The extra dimensions have a natural interpretation. Just as gravitation is the curvature of space-time, the other forces appear here as curvatures in the higher dimensions. Since a particle is Lie algebra valued form, it may, or may not have components in these extra dimensions. If it does, it feels the curvature and reacts accordingly. If it does not extend into these dimensions, it does not feel the forces.

The quantization scheme introduced in Section 3 is pure geometry. This was Einstein's vision—to obtain quantum theory from the geometry. The interaction of Δ, which generates the mass spectrum, with the Δ_ω, which generate the other quantum numbers could possibly explain particle generations. Only further research will tell.

The idea of obtaining supersymmetries from the representation on $\Lambda(M, G')$ appears to be new. Our approach could be summarized as "gauge, then super," whereas the standard approach is to "super then gauge." The standard approach adds an arbitrary number of super generators to the standard generators. Again only time and further research will tell the full story.

There is a C^* algebra lurking in the background. Using a normalized Hodge star operator as the * operator for the C^* algebra, we can consider

[1] For a current review see Hughston and Ward (1979).

the C^* algebra generated by $*$, d, and δ_ω for admissible ω. So even if $SU(3,2)$ is the correct group to gauge, there are many more questions left than have been answered. That is one of the greatest joys of research!

Obviously, the next step is to generate all the numbers that this theory is capable of producing. Hopefully I will report on these numbers at the Third New Orleans Conference on Quantum Theory and Gravitation. But there is one prediction immediate from the theory.

In the $SU(5)$ theory, the "extra" generators tacked onto $SU(3)\times SU(2)\times U(1)$ are compact. The presence of these compact generators leads to the necessity of introducing superheavy particles which interact via exotic forces which lead to proton decay. The extra generators in $SU(3,2)$ are not compact, do not lead to superheavy particles nor exotic forces (unless gravitation is considered exotic). Without this mechanism there is no reasonable way for protons to decay. So in the $SU(3,2)$ gauge theory, protons are stable.

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